2.7 Matched Filter

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Aim and Applications of the Matched Filter

♦ The matched filter is a linear filter that is designed to detect the presence of a waveform (or pulse) \( x(t) \) of known structure buried in additive noise. i.e. within a signal

\[
v(t) = Ax(t - t_0) + n(t)
\]

where \( A \) and \( t_0 \) are unknown constants, and \( n(t) \) is noise.

♦ The output of a matched filter will typically exhibit a sharp peak in response to the presence of the desired waveform (or pulse) at its input. From this peak may be determined time location \( t_0 \), and amplitude \( A \) of the pulse within the received waveform.

♦ E.g. 1. Radars transmit electromagnetic pulses which reflect off distance objects (or “targets”). The echoes are captured by a receiving antenna. The echoes may be very weak signals buried in additive noise (picked up by the antenna, and also created by the receiver itself). Matched filters are used to process the received waveform, and hence optimize the chance of detecting a target.

♦ E.g. 2. In telecommunications systems using pulse code modulation, a bank of \( N \) filters may be constructed, with each filter specifically designed to respond to the presence of a particular pulse code. Comparison of the responses at the outputs identifies the received code.
Model of Transmitted and Received Signal with Additive Noise

- Transmitted pulse \( x(t) \)
- Received signal \( v(t) = A x(t-t_0) + n(t) \)

where \( t_0 \) is the propagation time delay, 
\( A \) is an amplitude scaling factor, and 
\( n(t) \) is additive noise, described by its PSD \( S_n(\omega) \).

- The SNR of the received pulse is defined as the ratio of the peak signal power to average noise power, i.e.

\[
SNR = \frac{\text{peak signal power}}{\text{average noise power}} = \frac{\max \{|A x(t-t_0)|^2\}}{|n(t)|^2} = \frac{|A x(0)|^2}{|n(t)|^2}
\]

Definition of the Matched Filter

- Given an input signal \( v(t) = A x(t-t_0) + n(t) \)
  where \( x(t) \) is a pulse of known structure, 
  \( t_0 \) is an unknown delay, and \( A \) is an unknown scaling factor.

- A “matched filter” is a linear filter \( h(t) \) or \( H(\omega) \) that is designed to maximize the peak SNR at its output at some specified instant relative to \( t_0 \).

- The output is \( v_o(t) = v(t) \ast h(t) = y(t) + n_o(t) \)

- The aim is to find \( h(t) \) or equivalently \( H(\omega) \) that maximizes (at some instant \( t_x \), usually \( t_x > t_0 \)):

\[
\text{peak SNR} = \frac{|y(t_x)|^2}{|n_o(t)|^2}
\]
Derivation of the Matched Filter

Simplified Signal Model

Let the input be: \( v(t) = x(t) + n(t) \) (where \( x(t) \) is a known waveform plus stationary noise \( n(t) \))

Filter output: \( v_o(t) = v(t) \otimes h(t) = x(t) \otimes h(t) + n(t) \otimes h(t) \)

Let \( y(t) = x(t) \otimes h(t) \) (output signal component)

\( n_o(t) = n(t) \otimes h(t) \) (output noise component)

Output written as \( v_o(t) = y(t) + n_o(t) \)
Signal Model Illustrated

\[ v(t) = x(t) + n(t) \]

Output \[ v_o(t) = y(t) + n_o(t) \]

Maximize at \( t = t_d \)

Output \( SNR = \frac{|y(t_d)|^2}{|n_o(t)|^2} = \frac{\text{peak signal power}}{\text{average noise power}} \)

Derivation of the Matched Filter

Output (signal component)

\[ y(t) = x(t) \ast h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) H(\omega) e^{j\omega t} d\omega \]

(from inverse FT)

Peak output signal power

\[ |y(t_d)|^2 = \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) H(\omega) e^{j\omega t_d} d\omega \right|^2 \]

Average noise power

\[ |n_o(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{n_o}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_n(\omega)|H(\omega)|^2 d\omega \]
Optimization Problem

The objective is to find a/the filter $H(\omega)$ that optimizes the SNR at the time instant $t = t_d$:

$$\frac{|y(t_d)|^2}{|n_o(t)|^2} = \frac{1}{2\pi} \left| \int_{-\infty}^{+\infty} X(\omega) H(\omega) e^{j\omega t_d} d\omega \right|^2$$

To do this, we make use of a (provable) mathematical relationship called the Schwartz (or Cauchy-Schwartz) inequality.

Schwartz Inequality for Integrals

- The Schwartz inequality states that for two complex functions, $f(x)$ and $g(x)$, integrable over $[a,b]$

$$\left| \int_{a}^{b} f(x) g(x) \, dx \right|^2 \leq \int_{a}^{b} |f(x)|^2 \, dx \int_{a}^{b} |g(x)|^2 \, dx$$

- The equality holds (i.e. LHS = RHS) if $g(x) = k f^*(x)$ where $k$ is a real constant.

- If $g(x) \neq k f^*(x)$ then the left side is less than the right side.

Proof? Google it!
To maximize SNR, apply Schwartz Inequality

\[
\text{SNR: } \frac{|y(t_d)|^2}{|n_o(t)|^2} = \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)H(\omega)e^{j\omega t_d} d\omega \right|^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_n(\omega)|H(\omega)|^2 d\omega
\]

Rewrite expression as:

\[
\frac{|y(t_d)|^2}{|n_o(t)|^2} = \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{X(\omega)}{\sqrt{S_n(\omega)}} \left| \sqrt{S_n(\omega)}H(\omega)e^{j\omega t_d} \right| d\omega \right)^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_n(\omega)|H(\omega)|^2 d\omega
\]

Apply Schwartz Inequality with: 
\[ f(x) \equiv \left[ \frac{X(\omega)}{\sqrt{S_n(\omega)}} \right] \quad \text{and} \quad g(x) \equiv \sqrt{S_n(\omega)}H(\omega)e^{j\omega t_d} \]

Notice that the term on the right of the numerator cancels with the denominator.

\[
\frac{|y(t_d)|^2}{|n_o(t)|^2} \leq \frac{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{X(\omega)}{\sqrt{S_n(\omega)}} \right|^2 d\omega \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \sqrt{S_n(\omega)}H(\omega)e^{j\omega t_d} \right|^2 d\omega}{\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_n(\omega)|H(\omega)|^2 d\omega}
\]

The term on the right of the numerator cancels with the denominator.

\[
\frac{|y(t_d)|^2}{|n_o(t)|^2} \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{X(\omega)}{S_n(\omega)} \right|^2 d\omega \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_n(w)|H(\omega)|^2 d\omega \]

Swartz inequality becomes an “equality” i.e. LHS = RHS if 
\[ g(x) = k f^*(x) \]

or in our case, if:

\[ \sqrt{S_n(\omega)}H(\omega)e^{j\omega t_d} = k \left[ \frac{X(\omega)}{\sqrt{S_n(\omega)}} \right]^* \]

\[ \Rightarrow H(\omega) = k \frac{X(\omega)^*}{S_n(\omega)} e^{-j\omega t_d} \]
Matched Filter

The filter that maximizes the peak SNR at time \( t_d \) is:

\[
H(\omega) = k \frac{X^* (\omega)}{S_n (\omega)} e^{-j\omega t_d}
\]

\( k \) is a real constant, usually positive, and is often set to 1.

For the special case of white noise, \( S_n (\omega) = \text{constant} \)

\[
H(\omega) = k X^* (\omega) e^{-j\omega t_d}
\]

Inverse transforming gives:

\[
h(t) = k x^* (-t + t_d)
\]

Apply Fourier: \( X^* (\omega) \rightarrow x^* (-t) \)

and \( Y(\omega) e^{-j\omega t_d} \rightarrow y(t-t_d) \)

Matched Filter (additive white noise)

\( x(t) \)

\( x(-t) \)

\( h(t) = x^* (-t + t_d) \)

“Flip \( h(t) \), slide, mult. and integrate”

\( y(t) = x(t) \otimes h(t) \approx c^2 2B \text{Sa}(2\pi B(t-t_d)) \)
Interpretation in Frequency Domain:

\[ H(\omega) = k \frac{X^*(\omega)}{S_n(\omega)} e^{-j\omega t_d} \]

In the frequency domain, the matched filter magnitude boosts the frequency components for which the signal is large compared to the noise and suppresses those that are weak compared to the noise i.e.

\[ |H(\omega)| \propto \frac{|X(\omega)|}{S_n(\omega)} \]

The phase of the filter cancels the phase of \( X(\omega) \), \( H(\omega) \propto X^*(\omega) \) and hence the inverse Fourier transform integral gives a strong peak at \( t = t_d \).

In the white noise case

\[ H(\omega) = X^*(\omega) e^{-j\omega t_d} \]

\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) H(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 e^{-j\omega t_d} e^{j\omega t} d\omega \]

\[ y(t_d) = E_x \quad \text{(by Parseval)} \]

Interpretation in Time Domain

- In the time domain, the matched filter for white noise is seen as the conjugate of a time-reversed (“flipped”) version of \( x(t) \).
- It is easiest to consider the case of \( k=1 \) and \( t_d = 0 \), which is a matched filter with “zero-delay”. \( h(t) = x^*(-t) \)
- Performing a graphical convolution, “flips” \( h(t) \), which re-flips \( x^*(-t) \) such that it will now match (or correlate well) with the “\( x(t) \)” contained in the received signal \( v(t) \) as one slides from left to right and integrates the product.

\[ v(t) \ast h(t) = x(t) \ast h(t) + n(t) \ast h(t) \]

- The “width” of the peak in the time domain depends on the bandwidth \( B \) of \( X(\omega) \). To get \( y(t) \), we inverse transform:

\[ X(\omega) H(\omega) \propto |X(\omega)|^2 \]
Interpretation in Time Domain:

- The width of the peak in the time domain depends on the bandwidth $B$ of $X(\omega)$, since we are inverse transforming $|X(\omega)|^2$.
- If $|X(\omega)|^2$ has an approximately rectangular bandwidth, then the output will have a Sa( ) shape.
- The “3dB” width of the lobe will be $\delta t \approx \frac{1}{2B}$ [s].

\[ \delta t \approx \frac{1}{2B} \]

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Bandlimited input signal $x(t)$
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Matched Filter $H(\omega)$
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Output
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\[ \delta t \approx \frac{1}{2B} \]
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Physically Realizable Filters

- The impulse response of the matched filter for waveform $x(t)$ with additive white noise is $h(t) = x^*(-t + t_d)$.
- If this filter is to be constructed as a filter operating in “real-time”, then the output cannot respond before the input arrives.
- Thus for “physical realizability”, the impulse response $h(t)$ must begin after $t = 0$. i.e. $h(t) = 0$ for $t \leq 0$.
- In a real matched filter (built either with passive LRC components or using A/D, D/A and DSP technology), the MF peaks after the “entire waveform” has been fed into the input.
- In designing a filter, one can always make it physically realizable by shifting $h(t)$ to the right (by increasing $t_d$). This will simply delay the output response by the corresponding amount.
Relationship to Correlation

- Feeding a signal $v(t)$ into a matched filter $h(t) = x^*(-t + t_d)$ matched to waveform $x(t)$ (with white noise), is equivalent to a correlation operation between $x(t)$ and $v(t)$.

- Recall correlation for energy signals: $R_{vx}(\tau) = \int_{-\infty}^{\infty} v^*(t)x(t+\tau)dt$

- The matched filter output is: $\nu_o(t) = v(t) \otimes h(t)$

\[
\nu_o(t) = \int_{t'=-\infty}^{\infty} v(t')h(t-t')dt' = \int_{-\infty}^{\infty} v(t')x^*(t'+t_d-t)dt' \\
\text{put } u = t' + t_d - t
\]

\[
\nu_o(t) = \int_{u=-\infty}^{\infty} x^*(u)v(u+[t-t_d])du = R_{xv}(t-t_d)
\]

Relationship to Correlation cont...

- Thus if one correlates $x(t)$ and $v(t)$, and shifts the result to the right by an amount $t_d$, one gets the same output as from the matched filter operation.

- The matched filter output is $\nu_o(t) = v(t) \otimes h(t)$

\[
\nu_o(t) = R_{vx}(t-t_d)
\]

- Note: if $v(t) = x(t)$ (the case where the input matches perfectly) then the MF output is $\nu_o(t) = R_x(t-t_d)$ which is just the autocorrelation function of $x(t)$, shifted by $t_d$. 

Output SNR

- The output SNR of the matched filter is (previously derived general case)

\[
\frac{|y(t_d)|^2}{|n_0(t)|^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{X(\omega)}{S_n(\omega)} \right|^2 d\omega
\]

- For white noise case, \( S_n(\omega) = \eta/2 \) \( H(\omega) = X^*(\omega) e^{-j\omega t_d} \)

\[
|y(t_d)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t_d} d\omega = E_x^2
\]

\[
|n_0(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta |H(\omega)|^2 d\omega = \frac{\eta}{2} E_x = \frac{\eta}{2} E_x
\]

\[
\Rightarrow \frac{|y(t_d)|^2}{|n_0(t)|^2} = \frac{E_x^2}{\eta/2} = \frac{E_x}{\eta/2}
\]

- Thus the output SNR is proportional to the energy of the received signal (and hence the energy of the transmitted signal).

Example of Matched Filter: detection of a received pulse

**Transmitted pulse.** \( x(t) = A \text{ rect} \left( \frac{t-T/2}{T} \right) \)

**Matched Filter.** \( h(t) = x^*(t) \)

**Delayed, scaled pulse.** \( kx(t-t_0) \)

**Noisy received waveform.** \( v(t) = kx(t-t_0) + n(t) \)

**Output of matched filter; \( h(t) \otimes v(t) \)**

**SNR at peak is optimised.**

Note: output peak is proportional to energy in pulse \((A^2T)\)
Example of Matched Filter: detection of a received pulse

Transmitted pulse.
\[ x(t) \]

Matched Filter.
\[ h(t) = x^*(-t) \]

Delayed, scaled pulse.
\[ kx(t-t_0) \]

Noisy received waveform.
\[ v(t) = kx(t-t_0) + n(t) \]

Output of matched filter; SNR at peak is optimised.
\[ h(t) \otimes v(t) \]

Note: output peak is proportional to energy in pulse

Applications of Matched Filter

The matched filter is very useful for detecting known waveforms in the presence of additive noise.

E.g. 1. Radars use correlation receivers to detect weak echoes from distant targets. A replica of the transmitted pulse is correlated with the received waveform. The radar resolution (ability to separate close targets) depends on the bandwidth of the transmitted pulse.

E.g. 2. In telecommunications systems using pulse amplitude modulation, a matched filter, matched to the shape of the pulse, can be used to estimate optimally the pulse amplitude, since the height of the peak at the output of the matched filter is proportional to the amplitude of the received pulse.